

ON THE DETERMINATION OF STAR BODIES FROM THEIR HALF-SECTIONS

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ABSTRACT. We obtain explicit inversion formulas for the Radon-like transform that assigns to a function on the unit sphere the integrals of that function over hemispheres lying in lower dimensional central cross-sections. The results are applied to determination of star bodies from the volumes of their central half-sections.

1. INTRODUCTION

Let K be a compact subset of \mathbb{R}^n which contains the origin o as an interior point and is star-shaped with respect to o . Such a set K is called a *star body* if the radial function

$$\rho_K(\theta) = \max\{r \geq 0 : r\theta \in K\}, \quad \theta \in S^{n-1}, \quad (1.1)$$

that determines the shape of K is continuous; see Gardner [2]. We denote by $G_{n,k}$, $2 \leq k \leq n-1$, the Grassmann manifold of all k -dimensional linear subspaces ξ of \mathbb{R}^n . Passing to spherical coordinates, one can evaluate the k -dimensional volume of $K \cap \xi$ as

$$\text{vol}_k(K \cap \xi) = \frac{1}{k} \int_{S^{n-1} \cap \xi} \rho_K^k(\theta) d_\xi \theta, \quad (1.2)$$

where $d_\xi \theta$ stands for the corresponding $(k-1)$ -dimensional surface element. The right-hand side of (1.2) is a constant multiple of the Funk-type transform

$$(F_k f)(\xi) = \int_{S^{n-1} \cap \xi} f(\theta) d_\xi \theta, \quad \xi \in G_{n,k}, \quad (1.3)$$

that can be explicitly inverted by different ways provided that f is an even integrable function on S^{n-1} ; see, e.g., [7, 8, 11]. This fact makes it possible to determine the shape of K from the knowledge of the volumes of $K \cap \xi$ for all $\xi \in G_{n,k}$ provided that K is origin-symmetric.

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The star bodies which are not origin-symmetric cannot be uniquely reconstructed from the integrals (1.2); see, e.g., Gardner [2, Section 7.2]. This statement agrees with a known fact that the kernel of F_k on $L^1(S^{n-1})$ consists of odd functions.

In the case $k = n - 1$, Groemer [6] who followed some ideas from Backus [1], suggested to replace the hyperplane central sections by the “half-sections” $K \cap H(u, v)$, where

$$H(u, v) = \{x \in \mathbb{R}^n : x \cdot u = 0, x \cdot v \geq 0\}, \quad u \perp v, \quad (1.4)$$

is a half-plane determined by mutually orthogonal unit vectors u and v , so that the origin o lies on the boundary of $H(u, v)$. Thus (1.2) is substituted by the lower dimensional hemispherical integral

$$\text{vol}_{n-1}(K \cap H(u, v)) = \frac{1}{n-1} \int_{S^{n-1} \cap H(u, v)} \rho_K^{n-1}(\theta) d_\xi \theta. \quad (1.5)$$

One of the main results of [6] states that if the star bodies K and L satisfy

$$\text{vol}_{n-1}(K \cap H(u, v)) = \text{vol}_{n-1}(L \cap H(u, v))$$

for all $u, v \in S^{n-1}$, $u \perp v$, then $K = L$. This uniqueness result was extended by Goodey and Weil [5] to half-sections of arbitrary dimension $2 \leq k \leq n - 1$.

To the best of our knowledge, the following important questions remained open.

Question 1. *How can we explicitly reconstruct the shape of K (or the radial function ρ_K) from volumes (1.5) or, more generally, from the corresponding k -dimensional volumes?*

Question 2. *How can we eliminate overdeterminedness of the inversion problem in Question 1?*

Regarding Question 2, we observe that (1.4) parameterizes the corresponding half-sections by the elements of the Stiefel manifold $V_{n,2} = \{(u, v) : u, v \in S^{n-1}, u \perp v\}$, so that $\dim V_{n,2} = 2n - 3 > n - 1 = \dim S^{n-1}$ if $n > 2$. In other words, the dimension of the target space is greater than the dimension of the source space. The latter means that the inversion problem in Question 1 can be overdetermined. In the case $k < n - 1$, the difference between the dimensions of the target space and source space is bigger because the corresponding lower dimensional Funk transform (1.3) is overdetermined itself. This situation is pursuant to Gel’fand’s celebrated question [3] on how to reduce overdeterminedness of transformations in integral geometry. In our case it means the following

Problem. Find an $(n - 1)$ -dimensional submanifold \mathfrak{M} of the manifold of all k -dimensional central half-sections of K so that K could be recovered from the volumes of the half-sections belonging to \mathfrak{M} only.

In the present article we solve this problem give and an answer to Questions 1 and 2. The basic idea is to consider half-sections lying in the open half-spaces

$$H_{\pm} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \pm x_n > 0\} \quad (1.6)$$

separately. The case of all $2 \leq k \leq n - 1$ is considered in Section 2. Here our inversion formulas remain overdetermined if $k < n - 1$. This overdeterminedness is eliminated in Section 3.

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2. INVERSION FORMULAS

We consider the following hemispherical modifications of the Funk-type transform (1.3) when a function $f \in L^1(S^{n-1})$ is integrated over $(k - 1)$ -dimensional hemispheres $S_{\pm}^{n-1} \cap \xi$, $\xi \in G_{n,k}$, lying in the $(n - 1)$ -dimensional hemispheres

$$S_{\pm}^{n-1} = \{\theta = (\theta_1, \dots, \theta_n) \in S^{n-1} : \pm \theta_n > 0\},$$

respectively. Specifically, for any subspace $\xi \in G_{n,k}$, not orthogonal to $e_n = (0, \dots, 0, 1)$, we set

$$(F_k^+ f)(\xi) = \int_{S_+^{n-1} \cap \xi} f(\theta) d_{\xi} \theta \quad \left(= \int_{S_-^{n-1} \cap \xi} f(-\theta) d_{\xi} \theta \right), \quad (2.1)$$

$$(F_k^- f)(\xi) = \int_{S_-^{n-1} \cap \xi} f(\theta) d_{\xi} \theta \quad \left(= \int_{S_+^{n-1} \cap \xi} f(-\theta) d_{\xi} \theta \right). \quad (2.2)$$

Clearly,

$$(F_k^+ f)(\xi) = \frac{1}{2}(F_k f_1)(\xi), \quad f_1(\theta) = \begin{cases} f(\theta) & \text{if } \theta \in S_+^{n-1}, \\ f(-\theta) & \text{if } \theta \in S_-^{n-1}, \end{cases} \quad (2.3)$$

$$(F_k^- f)(\xi) = \frac{1}{2}(F_k f_2)(\xi), \quad f_2(\theta) = \begin{cases} f(-\theta) & \text{if } \theta \in S_+^{n-1}, \\ f(\theta) & \text{if } \theta \in S_-^{n-1}, \end{cases} \quad (2.4)$$

where f_i ($i = 1, 2$) are integrable even functions on S^{n-1} that can be reconstructed from $\varphi^{\pm} = F_k^{\pm} f$ by the formulas $f_1 = 2F_k^{-1} \varphi^+$ and

$f_2 = 2F_k^{-1}\varphi^-$. Combining these formulas, we obtain

$$f(\theta) = \begin{cases} 2(F_k^{-1}\varphi^+)(\theta) & \text{if } \theta \in S_+^{n-1}, \\ 2(F_k^{-1}\varphi^-)(\theta) & \text{if } \theta \in S_-^{n-1}. \end{cases} \quad (2.5)$$

Thus we have proved the following

Theorem 2.1. *For $2 \leq k \leq n-1$, a function $f \in L^1(S^{n-1})$ can be recovered from the integrals $\varphi^\pm = F_k^\pm f$ by the formula (2.5).*

Some comments are in order.

1. Let $\xi \in G_{n,k}$ be a subspace which is not orthogonal to $e_n = (0, \dots, 0, 1)$. We denote $\xi_\pm = \xi \cap \{x \in \mathbb{R}^n : \pm x_n > 0\}$. Then Theorem 2.1 implies the following

Corollary 2.2. *The radial function ρ_K of the star body K can be recovered from the volumes $v^\pm(\xi) = \text{vol}_k(K \cap \xi_\pm)$ by the formula*

$$\rho_K^k(\theta) = \begin{cases} 2k(F_k^{-1}v^+)(\theta) & \text{if } \theta \in S_+^{n-1}, \\ 2k(F_k^{-1}v^-)(\theta) & \text{if } \theta \in S_-^{n-1}. \end{cases} \quad (2.6)$$

For $\theta_n = 0$ it can be determined from (2.6) by continuity.

2. In the case $k = n-1$, we can set $\xi = u^\perp$, $u \in S^{n-1}$, and write (1.3) as the usual Funk transform

$$(Ff)(u) = \int_{\{\theta \in S^{n-1} : u \cdot \theta = 0\}} f(\theta) d_u \theta. \quad (2.7)$$

Similar notations $(F^\pm f)(u)$ can be used for the hemispherical transforms (2.1) and (2.2).

3. The above reasoning shows that to reconstruct f on S_+^{n-1} (or S_-^{n-1}) the knowledge of $\varphi^+ = F_k^+ f$ (or $\varphi^- = F_k^- f$, resp.) is sufficient.

4. A theory of the Funk transform provides a variety of inversion formulas for F_k ; see, e.g., [4, 7, 8, 9, 11]. The functions f_1 and f_2 in (2.3) and (2.4) can have discontinuity on the equator $\theta_n = 0$. It means that we cannot apply inversion formulas for F_k (at least, straightforward) in which the smoothness is crucial. However, if $f \in L^p(S^{n-1})$, $1 \leq p < \infty$, then f_1 and f_2 also belong to $L^p(S^{n-1})$ and can be reconstructed, e.g., by the method of mean value operators as follows.

For $r = \cos \psi$, $\psi \in [0, \pi/2]$, and $\theta \in S^{n-1}$, consider the shifted dual transform

$$(F_{k,\theta}^* \varphi)(r) = \int_{d(\theta, \xi) = \psi} \varphi(\xi) d\mu(\xi), \quad (2.8)$$

where φ is a function on $G_{n,k}$, $d(\cdot, \cdot)$ denotes the geodesic distance, and $d\mu(\xi)$ stands for the corresponding canonical measure; see [9, Section 5] for details.

Theorem 2.3. [9, Theorem 5.3] *An even function $f \in L^p(S^{n-1})$, $1 \leq p < \infty$, can be recovered from $\varphi = F_k f$ by the formula*

$$f(\theta) = \lim_{s \rightarrow 1} \left(\frac{1}{2s} \frac{\partial}{\partial s} \right)^k \left[\frac{\pi^{-k/2}}{\Gamma(k/2)} \int_0^s (s^2 - r^2)^{k/2-1} (F_{k,\theta}^* \varphi)(r) r^k dr \right]. \quad (2.9)$$

In particular, for k even,

$$f(\theta) = \lim_{s \rightarrow 1} \frac{1}{2\pi^{k/2}} \left(\frac{1}{2s} \frac{\partial}{\partial s} \right)^{k/2} [s^{k-1} (F_{k,\theta}^* \varphi)(s)]. \quad (2.10)$$

Alternatively,

$$f(\theta) = \lim_{s \rightarrow 1} \left(\frac{\partial}{\partial s} \right)^k \left[\frac{2^{-k} \pi^{-k/2}}{\Gamma(k/2)} \int_0^s (s^2 - r^2)^{k/2-1} (F_{k,\theta}^* \varphi)(r) dr \right]. \quad (2.11)$$

The limit in these formulas is understood in the L^p -norm.

3. THE CASE $k < n - 1$

If $k < n - 1$, the inversion problem for F_k^\pm and F_k is overdetermined because the dimension of the target space is greater than the dimension of the source space:

$$\dim G_{n,k} = k(n - k) > n - 1 = \dim S^{n-1}.$$

Below we eliminate this overdeterminedness by choosing an $(n - 1)$ -dimensional submanifold \mathfrak{M} of $G_{n,k}$ which is sufficient to reconstruct f from $(F_k^\pm f)(\xi)$ with $\xi \in \mathfrak{M}$. We proceed as in [10].

Suppose that $\{e_1, \dots, e_n\}$ is a standard orthonormal basis in \mathbb{R}^n and denote

$$\mathbb{R}^{n-k} = \mathbb{R}e_1 \oplus \dots \oplus \mathbb{R}e_{n-k}, \quad \mathbb{R}^k = \mathbb{R}e_{n-k+1} \oplus \dots \oplus \mathbb{R}e_n, \quad (3.1)$$

$$\mathbb{R}^{k+1} = \mathbb{R}e_{n-k} \oplus \mathbb{R}^k, \quad S^k = S^{n-1} \cap \mathbb{R}^{k+1}, \quad S_\pm^k = S_\pm^{n-1} \cap \mathbb{R}^{k+1}. \quad (3.2)$$

Given a point $v \in S^{n-k-1} = S^{n-1} \cap \mathbb{R}^{n-k}$, we fix an orthogonal transformation γ_v in \mathbb{R}^{n-k} , so that $\gamma_v e_{n-k} = v$. Let

$$\tilde{\gamma}_v = \begin{bmatrix} \gamma_v & 0 \\ 0 & I_k \end{bmatrix}, \quad (3.3)$$

where I_k is the identity $k \times k$ matrix. We denote by $G_k(\tilde{\gamma}_v \mathbb{R}^{k+1})$ the Grassmannian of all k -dimensional linear subspaces of $\tilde{\gamma}_v \mathbb{R}^{k+1}$ and set

$$\mathfrak{M} = \bigcup_{v \in S^{n-k-1}} G_k(\tilde{\gamma}_v \mathbb{R}^{k+1}). \quad (3.4)$$

The restrictions $\tilde{F}_k^\pm f$ of $F_k^\pm f$ onto \mathfrak{M} can be identified with functions on the set $\tilde{S}_{n,k} = \{(v, w) : v \in S^{n-k-1}, w \in S^k\}$. These functions are represented by the hemispherical integrals

$$(\tilde{F}_k^\pm f)(v, w) = \int_{\{\eta \in S_\pm^k : \eta \cdot w = 0\}} f_v(\eta) d_w \eta, \quad f_v(\eta) = f(\tilde{\gamma}_v \eta), \quad (3.5)$$

over $(k-1)$ -dimensional hemispheres on S^k . Thus \tilde{F}_k^\pm can be inverted as in the previous section.

Let us explain the details. We equip $\tilde{S}_{n,k}$ with the product measure $dv dw$, where dv and dw stand for the corresponding surface elements on S^{n-k-1} and S^k . Using [10, Theorem 3.2] and taking into account that the restrictions of f_v onto S_\pm^k and their even extensions (cf. (2.3), (2.4)) belong to the same L^p spaces, we obtain the following existence result.

Theorem 3.1. *Let $1 \leq k < n-1$, $f \in L^p(S^n)$, $n-k < p \leq \infty$. Then $(\tilde{F}_k^\pm f)(v, w)$ are finite for almost all $(v, w) \in \tilde{S}_{n,k}$. If $p \leq n-k$, then there are functions $\tilde{f}_\pm \in L^p(S^{n-1})$ for which $(\tilde{F}_k^\pm \tilde{f}_\pm)(v, w) = \infty$.*

Owing to (2.5), to reconstruct f from $\varphi_v^\pm = \tilde{\mathcal{F}}_k^\pm f$, it suffices to invert the usual co-dimension one Funk transform \tilde{F} on S^k . This gives

$$f_v(\eta) = \begin{cases} 2(\tilde{F}^{-1} \varphi_v^+)(\eta) & \text{if } \eta \in S_+^k, \\ 2(\tilde{F}^{-1} \varphi_v^-)(\eta) & \text{if } \eta \in S_-^k. \end{cases} \quad (3.6)$$

If f is a continuous function, its value at a point $\theta \in S^{n-1}$ can be found as follows. We interpret θ as a column vector $\theta = (\theta_1, \dots, \theta_{n+1})^T$ and set

$$\theta' = (\theta_1, \dots, \theta_{n-k})^T \in \mathbb{R}^{n-k}, \quad \theta'' = (\theta_{n-k+1}, \dots, \theta_n)^T \in \mathbb{R}^k,$$

Suppose that $\theta' \neq 0$ and set

$$v = \theta' / |\theta'| \in S^{n-k-1}, \quad \eta = (0, \dots, 0, |\theta'|, \theta'')^T \in S^k. \quad (3.7)$$

Then $\tilde{\gamma}_v \eta = \theta$ and we get $f(\theta) = (\mathcal{F}^{-1} \varphi_v)(\eta)$ for v and η as in (3.7). If $\theta' = 0$, then $f(\theta)$ can be reconstructed by continuity from its values at the neighboring points.

If f is an arbitrary function in $L^p(S^n)$, $n-k < p \leq \infty$, then f can be explicitly reconstructed at almost all points on almost all spheres

$S_v^k = \tilde{\gamma}_v S^k$ by making use of known inversion formulas for the Funk transform on this class of functions; see, e.g., [10, p. 297].

The above reasoning is obviously applicable to reconstruction of the radial function ρ_K of the star body K from the volumes $v^\pm(\xi) = \text{vol}_k(K \cap \xi_\pm)$ with $\xi \in \mathfrak{M}$, as in Corollary 2.2.

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